Announcements

1) 3-4 2090 CB, Tomoki Ohsawa

2) HW due Thursday

3) Scholarship App online submission

Definition: (compactness)

A subset S of a metric space X is compact if

$$(x_n)_{n=1}^{\infty} \subseteq S \Rightarrow (x_n)_{n=1}^{\infty}$$

has a convergent subsequence whose limit is in S.

Idea: (size) A compact set is somehow "small".

Theorem: (Heine-Borel)
In $\mathbb{R}$ (or $\mathbb{R}^n$) with the usual metric, a subset S is compact if and only if it is closed and bounded.

proof $\Rightarrow$ Suppose S is compact. We need to show S is closed and bounded.

S closed: Let x be a limit point of S. We want x to be in S.

Since x is a limit point,

$\exists \ (x_n)_{n\in\mathbb{N}} \subseteq S$, $x_n \neq x \ \forall n\in\mathbb{N}$,

$$\boxed{\lim_{n\to\infty} x_n = x,}$$

Since S is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ whose $\boxed{\text{limit is in } S}$

But $(x_n)_{n\in\mathbb{N}}$ is convergent,

so since x_n converges to

x, x_{n_k} converges to x.

By compactness, $x \in S$.

This shows S is closed.

<u>S is bounded</u> By contradiction!

Suppose S is unbounded

Then $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in S$,

$$|x_n| > n.$$

Then $(x_n)_{n \in \mathbb{N}}$ can have no convergent subsequence, as $|x_{n_k}| > n_k$

Therefore, S is not compact, contradiction. Then S must be bounded.

We've shown: S compact $\Rightarrow$ S closed + bounded

This direction holds in any metric space

$\Leftarrow$ Suppose S is closed and bounded.

Let $(x_n)_{n \in \mathbb{N}}$ be any sequence in S. Since S is bounded, $(x_n)_{n \in \mathbb{N}}$ is bounded, so by Bolzano-Weierstrass, $(x_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$.

Suppose $\lim_{k \to \infty} x_{n_k} = x$

If $x \in S$, then done

However, $x \notin S \Rightarrow$

x is a limit point

of S. But since

S is closed, $x \in S$,

contradiction.

Therefore $x \in S$

and S is compact. $\square$

Example 1: (compact in $\mathbb{R}$)

$$[0,1] \cup [2,3]$$

is compact since it is bounded (by 3) and is the union of 2 closed intervals, hence it is closed.

Warning! In a general metric space, there is no Bolzano-Weierstrass theorem It is possible to have closed and bounded sets in this situation that are not compact (see HW)

Another Equivalent Formulation

Definition: (open cover)

Let S be a subset of a metric space X. An open cover of S is a collection of open sets $\{O_i\}_{i \in I}$ (I an index set) with

$$\boxed{S \subseteq \bigcup_{i \in I} O_i}$$

Example 2: If $S \subseteq \overline{X}$,

let $\varepsilon > 0$ be fixed and

let $O_x = B(x, \varepsilon)$.

Then if $I = \{x \in S\}$,

$$S \subseteq \bigcup_{x \in I} B(x, \varepsilon)$$

since $\forall\ x \in S$, $x \in B(x, \varepsilon)$

Could also write as

$$S \subseteq \bigcup_{x \in S} B(x, \varepsilon)$$

Theorem: X a metric space and $S \subseteq X$, then S is compact if and only if for every open cover $\{O_i\}_{i \in I}$ of S, $\exists$ $n \in \mathbb{N}$ and $O_{i_1}, O_{i_2}, \ldots, O_{i_n} \in \{O_i\}_{i \in I}$ with

$$S \subseteq \bigcup_{j=1}^{n} O_{i_n}$$

"Every open cover admits a finite subcover."

Use the open cover
formulation on
question #4 in HW

Connectedness

If compact is "small,"

connectedness is

"one piece" or

"interlinked."

Intuitively think of what it means for a set not to be connected - either "two or more pieces" or "divided".

Definition: (separated, disconnected)

Two subsets S and T of a metric space X are separated if $\overline{S} \cap T = S \cap \overline{T} = \emptyset$.

A subset $Y \subseteq X$ is disconnected if $\exists\, S, T \subseteq Y$, S, T separated and $S \cup T = Y$